UNPUBLISHED PRELIMINARY DATA NsG-134

Sound Waves In Relativistic Magnetohydrodynamics

By

R. P. Kanwal

1. Introduction

The mathematical treatment of the coupled motion of hydrodynamic flow and electromagnetic fields has assumed great importance recently. In the study of non-relativistic magnetohydrodynamics it is customary to adopt only the quasi-equilibrium approximation to the electrodynamic equations in which the displacement currents and charge accumulations are ignored. Such an analysis merely allows the possibility that the locities, both random and ordered, of fluid elements are comparable th the classical velocity of sound. Recently various attempts have been ade to give a covariant formulation to magnetohydrodynamics in which the ove-mentioned approximations are not introduced. In this paper we are iterested in the motion of sonic discontinuities with such a covariant ormulation. Pioneering work in the field of wave motion has been done by e Hoffman and Teller [7], Reichel [11], Zumino [16], Akhieser and Polovin [1], Coburn [2,3], Bruhat [5], Saini [12], Giere [6] and others. There are some disagreements in various results obtained by these authors. We plan to compare and extend the salient features of their analysis in the following sections.

Relativistic hydrodynamics has been formulated and put on firm foundations by various authors, while the equations of electrodynamics lend themselves readily into covariant description. Therefore the coupling of these two fields becomes rather simple. Recently Thomas [15] and Edlen [4] have given a discussion of relativistic hydrodynamics and re-defined the equations governing the flow of a perfect relativistic gas.

We couple these modified equations with those of electrodynamics. The total energy tensor, thus obtained, enables us to compare various known results on wave motion. We consider only those fluids which have infinite conductivity.

We consider a weak disturbance whose propagation can be represented by a time-like hypersurface in an Einstein-Riemann space. By using the concept of singular surfaces across which all the magnetohydrodynamic quantities are continuous but the first derivatives of at least one of these quantities is discontinuous, we obtain various speeds of propagation of these surfaces. We then compare our results with the known results in this field. Furthermore, by taking a suitable metric we give a space-time representation of our analysis. This enables us to compare the present discussion with that known for non-relativistic magnetohydrodynamics. We find that qualitatively the present results are similar to the ones known for the non-relativistic case. In fact, we have three kinds of waves - slow, intermediate and fast. The possibility of any of these waves exceeding the speed of light is excluded since we have taken the wave normals to be space like.

The description of the propagation of these wave fronts is made more quantitative by employing the classical theory of ray optics. By representing the wave front as the surface of $f(x, t) = o$, the equations which give various wave speeds become first-order partial differential equations in the function f. These equations, in turn, have characteristics which are described as curves rather than cones. These curves are called bicharacteristics or rays. For each category of waves - slow, intermediate or fast - we have a direction field of rays along which the wave fronts propagate. We derive the equations for these rays. From the discussion of these equations it follows

[redacted]

that the bicharacteristics corresponding to the intermediate waves always lie in the plane determined by the velocity and magnetic 4 - vectors.

The geometrical shape of the wave fronts can be analysed with the help of the equations for the rays. However, it is simpler to discuss the geometry of wave fronts with the help of the surfaces of wave normals. We derive the equations for these surfaces for all the three waves. It turns out that, as in the non-relativistic case, the surface of the wave normals for the intermediate wave consist of two planes. The corresponding surfaces for slow and fast waves are two distinct nappes of a conoid; the surface of wave normals for the intermediate wave always touches this conoid.

2. Fundamental equations

Let g_{AB} denote the metric components of an Einstein-Riemann space (referred to hereafter as E-R space) and let W_A stand for the covariant components of the unit time-like velocity 4-vector $\underline{W}$. The capital Roman indices have the range 0, 1, 2, 3, and are subject to the summation convention over this range. Thus

$$g^{AB} W_A W_B = 1 . \qquad (2.1)$$

The momentum-energy tensor with components T_{AB} is composed of two parts $T^{(m)}_{AB}$ and $T^{(e)}_{AB}$.

The material part $T^{(m)}_{AB}$ is given as

$$T^{(m)}_{AB} = c^2 \rho W_A W_B - p g_{AB}, \qquad (2.2)$$

where ρ is the mass-energy density, p is the pressure and c is the velocity of light in vacuum. The components T_{AB} have the dimensions of energy.

The electromagnetic part $T^{(e)}_{AB}$ is defined for infinitely conducting fluids in terms of the electromagnetic skew-symmetric tensor with components H_{AB}, as

$$T^{(e)}_{AB} = \frac{1}{\mu} \left\{ \frac{1}{4} g_{AB} (H^{CD} H_{CD}) - H^{C}_{A} H_{CB} \right\}, \tag{2.3}$$

where the constant μ is the magnetic permeability. Let us now introduce the magnetic 4-vector $\underline{h}$ with components h_A [10]:

$$h_A = \frac{1}{\mu} \overset{*}{H}_{BA} W^B, \tag{2.4}$$

where the components $\overset{*}{H}_{AB}$ of the dual electromagnetic tensor are defined by the formulae

$$\overset{*}{H}_{AB} = \frac{1}{2} \eta_{ABCD} H^{CD}, \tag{2.5}$$

and η_{ABCD} are the components of the customary permutation tensor. In terms of these skew-symmetric tensors the Maxwell equations become

$$g^{AC} \overset{*}{H}_{AB;C} = 0, \qquad \frac{1}{\mu} H^{AB}_{\ ;A} = J^B, \tag{2.6}$$

where J^B are the components of the current 4-vector $\underline{J}$ and the semicolon denotes the covariant differentiation formed on g_{AB}. The electric 4-vector with components e_A is defined as

$$e_A = H_{BA} W^B. \tag{2.7}$$

In view of the skew-symmetry of H_{AB}, it readily follows that both the electromagnetic 4-vectors are orthogonal to $\underline{W}$:

$$h_A W^A = e_A W^A = 0. \tag{2.8}$$

Furthermore their components in a rest frame i.e. $W^o = 1$, $W^i = 0$, coincide with the magnetic and electric 3-vectors with components H_i and E_i. In these expressions as well as in the sequel, the Latin indices will have the range 1, 2, 3. For infinitely conducting fluids, it follows from Ohm's law (for a critical discussion see Coburn [2] and Reichel [11] that $e_A = 0$. Therefore the relation (2.7) becomes

$$H_{BA} W^B = 0. \tag{2.9}$$

This means that the 4-vector $\underline{W}$ lies in the null domain of the skew-symmetric matrix $((H_{AB}))$. Hence the rank of this matrix is less than four. But an antisymmetric matrix has always an even rank. Thus the rank of this matrix is two. From the definition of the dual tensor it follows that the matrix defined by the components $\overset{*}{H}_{AB}$ has also rank two and that the 4-vector $\underline{W}$ lies in its non-null domain. Furthermore since $\overset{*}{H}_{AB} h^B \neq o$, the vector $\underline{h}$ also lies in the non-null domain of this matrix. Since we have two mutually orthogonal vectors lying in the non-null domain of the skew-symmetric matrix $((\overset{*}{H}_{AB}))$ of rank two, we can decompose it into the bivector form [13]

$$\overset{*}{H}_{AB} = \mu (W_A h_B - W_B h_A). \tag{2.10}$$

where the coefficient μ is accounted for by the relation (2.4).

Hence

$$H_{AB} = \mu \, \eta_{ABCD} W^C h^D. \tag{2.11}$$

Substituting (2.10) into the first of the equations (2.6) we obtain an equation for the determination of the vector $\underline{h}$:

$$(h^A W^B - h^B W^A)_{;B} = 0 . \tag{2.12}$$

Once we have found $\underline{h}$, we recover the quantities H_{AB} and $\underline{J}$ from the equations (2.12) and the second equation (2.6). To evaluate $T^{(e)}_{AB}$ as given by the equation (2.3) we observe the following simple relation

$$H_{CD} H^{CD} = 2\mu^2 h_A h^A = -2\mu^2 \left| h \right|^2 , \tag{2.13}$$

where we have used the fact that $\underline{h}$, being orthogonal to the time-like vector $\underline{W}$, is a space-like 4-vector. Similarly

$$H^C_A H_{CB} = \mu^2 \cdot \left\{ (g_{AB} - W_A W_B) \left| h \right|^2 + h_A h_B \right\} . \tag{2.14}$$

Substituting (2.13) and (2.14) into the equation (2.3), we have

$$T^{(e)}_{AB} = \mu \left\{ (W_A W_B - \tfrac{1}{2} g_{AB}) \left| h \right|^2 - \tfrac{1}{2} h_A h_B \right\} , \tag{2.15}$$

which is a symmetric tensor. Let us observe in passing that

$$W_A \, T^{(e)\ AB}_{;\ B} = 0 = h_A \, T^{(e)\ AB}_{;B} . \tag{2.16}$$

We are now in a position to write down the complete value of T_{AB} in terms of two unknown vectors $\underline{W}$ and $\underline{h}$:

$$T_{AB} = c^2 \rho \, W_A W_B - p \, g_{AB} + \mu \left\{ (W_A W_B - \tfrac{1}{2} g_{AB}) \left| h \right|^2 - h_A h_B \right\} . \tag{2.17}$$

The Einstein field equations of general relativity imply that

$$T^{AB}_{;B} = 0 . \tag{2.18}$$

When we substitute (2.17) into (2.18) we get the required partial differential equations.

3. Definitive System of equations

Let us collect all the necessary equations and put them in the form needed in the subsequent analysis. In that connection the relation (2.1), when differentiated yields

$$W^A W_{A;B} = 0. \tag{3.1}$$

Similarly from the first relation (2.8) we get

$$W^A h_{A;B} + W^A_{;B} h_A = 0. \tag{3.2}$$

The field equations (2.12) are

$$W^B h^A_{;B} + h^A W^B_{;B} - h^B W^A_{;B} - W^A h^B_{;B} = 0. \tag{3.3}$$

The conservation law (2.18) becomes

$$\begin{aligned} &(c^2 \rho_{,B}) W^A W^B + (c^2 \rho + \mu |h|^2)(W^A W^A_{;B} + W^B W^A_{;B}) \\ &- p_{,B}\, g^{AB} - 2\mu(W^A W^B - \tfrac{1}{2} g^{AB}) h_c h^c_{;B} \\ &- \mu(h^A_{;B} h_B + h^A h_{B;B}) = 0. \end{aligned} \tag{3.4}$$

The thermodynamic relation, required for the present analysis, expresses the condition that [15]

$$p_{,A} W^A - a^2 \rho_{,A} W^A = 0, \tag{3.5}$$

where

$$a^2 = \frac{\gamma p}{\rho}, \tag{3.6}$$

and γ is a material constant. As we shall soon see that difference in various characteristic relations, obtained thus far, is accounted for by the difference in the thermodynamic relations. Except for the identity (3.2), we have now ten equations (we count (3.1) as one equation, derived as it is from a scalar equation (2.1)) in ten unknowns $\underline{W}$, $\underline{h}$, ρ and p.

4. Sonic disturbances

We consider a weak disturbance whose position can be represented by a three-dimensional time-like hypersurface $\Sigma(t)$, called a 3-wave, in the E-R space. Any time section t = constant, of Σ will be a two dimensional surface S(t), called the 2-wave, in the three-dimensional Riemann space R(t). Let N_A be the components of the unit space-like 4-vector $\underline{N}$ which is normal to $\Sigma(t)$. Furthermore let the bracket [F] stand for the jump of F across the sonic discontinuity. We assume that

$$[\rho] = [p] = [W_A] = [h_A] = [g_{AB}] = [g_{AB;C}] = [g_{AB;CD}] = 0. \tag{4.1}$$

When $[F] = 0$, but $[F_{;A}] \neq 0$, it can be shown that

$$[F_{;A}] = N_A \, \delta F, \tag{4.2}$$

where δF is the strength of the discontinuity. Moreover, since the components of the metric tensor and their first derivatives are continuous, it follows that the jump in the covariant derivatives of a quantity is equal to the jump in the ordinary derivatives. Also since the second derivatives of the components of the metric tensor are also continuous, it follows from the Einstein field equation that the quantities T^{AB} are continuous across the wave front.

Applying the jump condition (4.2) to the system of equations (3.1) - (3.5), there result the relations

$$W^A \delta W_A = 0; \tag{4.3}$$

$$W^A \delta h_A = - h_A \delta W^A; \tag{4.4}$$

$$L \delta h^A + h^A N_B \delta W^B - h_N \delta W^A - W^A N_B \delta h^B = 0; \tag{4.5}$$

$$c^2 L W^A \delta\rho + (c^2 \rho + \mu |h|^2)(W^A N_B \delta W^B + L \delta W^A) - N^A \delta p - \mu (2 L W^A - N^A) h_c \delta h^c$$
$$- \mu(\delta h^A h_N + h^A N_B \delta h^B) = 0, \tag{4.6}$$

and

$$\delta p = a^2 \delta \rho \tag{4.7}$$

where $L = W^A N_A$, and $h_N = h^A N_A$.

5. Alfven waves.

Let T_A stand for the components of the tangent 4-vector $\underline{T}$ to the wave front. We shall take such a tangential direction $\underline{T}$ that it is orthogonal to the 3-base $\underline{N}$, $\underline{W}$, and $\underline{h}$. If we multiply (4.5) and (4.6) by T_A and sum on the repeated index A, the following two relations result

$$L \delta h_T - h_N \delta W_T = 0, \tag{5.1}$$

and

$$(c^2 \rho + \mu |h|^2) L \delta W_T - \mu h_N \delta h_T = 0, \tag{5.2}$$

where $\delta h_T = \delta h^A T_A$ and $\delta W_T = \delta W^A T_A$. These two equations will give a non-trivial solution for $(\delta h_T, \delta W_T)$, if we have

$$L^2 (c^2 \rho + \mu |h|^2) - \mu h_N^2 = 0. \tag{5.3}$$

Let us now define

$$V = cL = cW^A N_A, \tag{5.4}$$

then (5.3) gives

$$V^2 = \frac{\mu\, h_N^2}{\rho + \mu |h|^2 / c^2} \quad . \tag{5.5}$$

This relation, as we shall soon see, leads to the Alfven wave of non-relativistic magnetohydrodynamics. It is independent of the thermodynamics in the above analysis. The same relation holds for incompressible as well as compressible fluids.

6. Fast and Slow Waves.

We start with the field equation (4.5). Multiplying it by W_A, summing on the repeated index A, and using (4.3) and (4.4), we derive

$$LW_A\, \delta h^A = N_B \delta h^B = - Lh_A \delta W^A \ . \tag{6.1}$$

Similarly if we multiply (4.5) by h_A, and sum on A, it yields

$$h_A \delta h^A = \frac{1}{L} \left(|h|^2\ N_B \delta W^B + h_N h_A \delta W^A \right). \tag{6.2}$$

Let us now replace δp with $a^2\ \delta\rho$ in the equation (4.6), as implied by the relation (4.7); multiply the resulting equation by h_A; use some of the other identities as derived above, it leads to the relation:

$$h_A \delta W^A = \frac{1}{L}\ \frac{a^2}{c^2}\ h_N\ \frac{\delta\rho}{\rho} \quad . \tag{6.3}$$

Similarly if we multiply the equation (4.6) with $\underline{W}$ and $\underline{N}$ and sum on the index A, we get

$$N_B \, \delta W^B = - L \left(1 - \frac{a^2}{c^2}\right) \frac{\delta \rho}{\rho} \; . \qquad (6.4)$$

$$(c^2 L^2 + a^2)\, \delta\rho + \frac{1}{L}(2L^2 c^2 \rho - \mu |h|^2) N_B \, \delta W^B - \frac{\mu}{L} h_N h_A \delta W^A = 0. \qquad (6.5)$$

Substituting from (6.3) and (6.4) into (6.5), we readily obtain

$$\left\{(2a^2 - c^2)\rho c^2 L^4 - ((a^2 - c^2)\mu |h|^2 - a^2 c^2 \rho) L^2 - a^2 \mu h_N^2\right\} \delta\rho = 0. \qquad (6.6)$$

But $\delta\rho \neq 0$; as such (6.6) gives

$$(c^2 - 2a^2)\rho c^2 L^4 - \left\{(c^2 - a^2)\mu |h|^2 + a^2 c^2 \rho\right\} L^2 + a^2 \mu h_N^2 = 0. \qquad (6.7)$$

In terms of V, as defined by (5.4), the above relation becomes

$$\left(1 - \frac{2a^2}{c^2}\right)\rho \; V^4 - \left\{\left(1 - \frac{a^2}{c^2}\right) \mu |h|^2 + a^2 \rho\right\} V^2 + a^2 \mu h_N^2 = 0 \qquad (6.8)$$

When $h_A = 0$, the equation (6.8) reduces to the one found by Thomas for uncharged fluids.

7. Comparison with known results.

If we take [14]

$$\rho = \rho_o \left(1 + \frac{e}{c^2} + \frac{p}{\rho_o c^2}\right), \qquad (7.1)$$

where ρ_o and e are the rest density and the rest internal energy respectively, the relations (5.3) and (5.7), agree with those found by Coburn [2].

To get the corresponding characteristic equations as derived by Bruhat [5], we set

$$\rho = \tilde{\rho} + p, \qquad c = 1, \qquad (7.2)$$

and take the thermodynamic relation, which gives

$$\delta p = \tilde{a}^2 \, \delta \, \tilde{\rho}, \qquad \tilde{a}^2 = \frac{dp}{d\tilde{\rho}} \, . \tag{7.3}$$

The modification in our analysis starts with the equation (3.4). It becomes

$$(\tilde{\rho} + p)_{,B} \, W^A W^B + (\tilde{\rho}+p+\mu|h|^2)(W^A W^B{}_{;B} + W^B W^A{}_{;B})$$

$$-p_{,B} g^{AB} - 2\mu(W^A W^B - \tfrac{1}{2} g^{AB}) h_C h^C{}_{;B} -$$

$$-\mu(h^A{}_{;B} h_B + h^A h_{B;B}) = 0 \quad . \tag{3.4)'$$

If we go through the same algebraic steps as with (3.4) and the related system of equations, the characteristic relations corresponding to (5.3) and (6.7) become

$$L^2 \, (\tilde{\rho} + p + \mu \, |h|^2 \,) = \mu \, h_N^2 = 0, \tag{5.3)'$$

and

$$(1-\tilde{a}^2)\,(\tilde{\rho}+p)L^4 - \left\{(\mu \, |h|^2 + \tilde{a} \, (\tilde{\rho} + p)\right\} L^2 + \tilde{a}^2 \, \mu \, h_N^2 = 0. \tag{6.7)''$$

These agree with the ones found by Bruhat.

Finally, if we take the relation (7.1) and assume the thermodynamic equation which leads to

$$\delta p = a_o^2 \, \delta\rho_o, \qquad a_o^2 = \frac{dp}{d\rho_o} \, . \tag{7.4}$$

The relation (5.3) is unchanged while (6.7) is modified into:

$$\left\{(c^2 + e+p/\rho_o) - a_o^2\right\}\rho_o(c^2+e+p/\rho_o)L^4$$

$$-\left\{(\rho_o a_o^2 + \mu|h|^2)(c^2 + e + p/\rho_o)\right\}L^2 + a_o^2\,\mu\, h_N^2 = 0, \qquad (6.7)''$$

which agrees with the result found by Saini [12] and Giere [6].

8. A space-time representation.

A coordinate system can be introduced in the E-R space for which the square of the element of length ds^2, has the form

$$ds^2 = c^2\, dt^2 - a_{ij}dx^i dx^j, \qquad (8.1)$$

where a_{ij} are the coefficients of a positive definite quadratic form. These coefficients, in general, depend on the coordinate t, as well as the spatial coordinates $\underline{x}$. Relative to this coordinate system the velocity 4-vector becomes

$$W_o = \frac{c}{\sqrt{1-v^2/c^2}}\ ;\ W_i = \frac{-v_i}{c\sqrt{1-v^2/c^2}}\ ,$$

$$W^o = \frac{1}{c\sqrt{1-v^2/c^2}}\ ;\ W^i = \frac{v^i}{c\sqrt{1-v^2/c^2}}\ , \qquad (8.2)$$

where v_i and v^i are the covariant and contravariant components of the velocity vector in the three-dimensional Riemann space R(t) whose metric is defined by the above quantities $a_{ij}(t,x)$, and v is the magnitude of this velocity.

The electromagnetic 4-vector $\underline{h}$ is now given as

$$h_o = \frac{(H_i v^i)}{\sqrt{1-v^2/c^2}} \quad , \quad h_i = -\frac{H_i}{\sqrt{1-v^2/c^2}} - \frac{(\underline{E} \times \underline{v})_i}{\mu c\sqrt{1-v^2/c^2}} \quad ,$$

$$h^o = \frac{(H_i v^i)}{c^2\sqrt{1-v^2/c^2}}, \quad h^i = \frac{H^i}{\sqrt{1-v^2/c^2}} + \frac{(\underline{E} \times \underline{v})^i}{\mu c\sqrt{1-v^2/c^2}} \tag{8.3}$$

Similarly the normal 4-vector $\underline{N}$ has the decomposition

$$N_o = \frac{G}{\sqrt{1-G^2/c^2}} \; ; \; N_i = \frac{-n_i}{\sqrt{1-G^2/c^2}}$$

$$N^o = \frac{G}{c^2\sqrt{1-G^2/c^2}} \quad ; \quad N^i = \frac{n^i}{\sqrt{1-G^2/c^2}} \quad , \tag{8.4}$$

where n_i and n^i are the covariant and contravariant components of the unit normal to S(t) in the space R(t) and G is the normal coordinate velocity of propagation of S(t) in R(t). The direction of $\underline{N}$ is chosen, for definiteness, so that the associated vector $\underline{n}$ is directed into the region of R(t) into which the surface S(t) is propagated; then the velocity G is positive.

The quantities L, V, h_N and $|h|^2$ become

$$L = N^A W_A = \frac{G - v_n}{c\sqrt{1-G^2/c^2}\;\sqrt{1-v^2/c^2}} \quad , \tag{8.5}$$

$$V = cL = \frac{G - v_n}{\sqrt{1-G^2/c^2}\,\sqrt{1-v^2/c^2}} \quad , \tag{8.6}$$

$$h_N = h_A N^A = \frac{(H_i v^i)G}{c^2\sqrt{1-G^2/c^2}} - \frac{H_n}{\sqrt{1-v^2/c^2}\,\sqrt{1-G^2/c^2}} - \frac{1}{\mu}\,\frac{(\underline{E} \times \underline{v})_i n^i}{c\sqrt{1-v^2/c^2}\,\sqrt{1-G^2/c^2}} \; , \tag{8.7}$$

and

$$|h|^2 = -h^A h_A = -\frac{(H_i v^i)^2}{c^2(1-v^2/c^2)} + \frac{H^2}{(1-v^2/c^2)} + \frac{2H^i(\underline{E} \times \underline{v})_i}{\mu c(1-v^2/c^2)} + \frac{|(\underline{E} \times \underline{v})|^2}{\mu^2 c^2(1-v^2/c^2)} \; . \tag{8.8}$$

From (8.6), (8.7) and (8.8) we observe the important fact that when

$$\frac{v}{c} \ll 1, \; \frac{G}{c} \ll 1,$$

we have

$$V \sim G - v_n \overset{\text{def.}}{=\!=} U, \tag{8.9}$$

$$h_N \sim - H_n, \tag{8.10}$$

$$|h|^2 \sim H^2 \; . \tag{8.11}$$

From (7.1) we observe that

$$\rho \sim \rho_o \, ,$$

and therefore a $\sim$ the velocity of sound in non-relativistic uncharged fluids.

When we make these substitutions in the characteristic relations (5.3) and (6.7), they become in the non-relativistic limit

$$U^2 = \mu H_n^2/\rho_o, \tag{8.12}$$

and

$$\rho_o U^4 - (\mu H^2 + a^2 \rho_o) U^2 + a^2 \mu H_n^2 = 0. \tag{8.13}$$

In fact they become obvious if we start with the equations (5.5) and (6.8), instead. These two relations agree completely with those obtained in non-relativistic magnetohydrodynamics [8] . The same is of course true of (5.3)', (6.7)' and (6.7)''. Qualitatively the results in the present case are similar to those in non-relativistic case. There are slow, intermediate and fast waves. The velocities of these waves are, however, comparable with the velocity of light but none can exceed that velocity. This follows from the Darmois-Lichnerowicz formula [9] for the speed u of wave propagation:

$$u^2 = \frac{c^2 \, W^A N_A}{W^A N_A - N^A N_A} \ . \tag{8.14}$$

Since $N^A N_A = -1$, we get that $u < c$.

Various other known results in connection with the characteristics can be deduced from the foregoing ones as special cases.

9. Bicharacteristics

The analysis in this section concerns the theory of rays as based on the characteristic relations (5.3) and (6.7). Let tne hypersurface $\Sigma(t)$ have an equation:

$$f(x,t) = 0, \tag{9.1}$$

then the components N_A are defined as

$$N_A = f_{,A}/\sqrt{|g^{CD} f_{,C} f_{,D}|}\ . \tag{9.2}$$

In terms of $f_{,A}$ we can write the characteristic relations (5.3) and (6.7) as

$$F(f_{,A}) = (W^A f_{,A})^2 (c^2 \rho + \mu |h|^2) - \mu (h^A f_{,A})^2 = 0, \tag{9.3}$$

and

$$F(f_{,A}) = (c^2 - 2a^2) \rho c^2 (W^A f_{,A})^4 + \left\{(c^2 - a^2)\mu |h|^2 + a^2 c^2 \rho\right\} (W^A f_{,A})^2 (g^{CD} f_{,C} f_{,D})$$
$$- a^2 \mu (h^A f_{,A})^2 (g^{CD} f_{,C} f_{,D}) = 0. \tag{9.4}$$

We have taken advantage of the relation $|g^{AB} f_{,A} f_{,B}| = (-g^{AB} f_{,A} f_{,B})$. The equations (9.3) and (9.4) are first order partial differential equations for determining the integral surface f. This integral surface is spanned by characteristic rays. These rays are called the bicharacteristics for the original equations and are determined by the ordinary differential equations

$$\frac{dx^A}{ds} = \frac{\partial F}{\partial (f_{,A})}\ , \tag{9.5}$$

with s defined as a parameter along the bicharacteristics. Substituting from (9.3) and (9.4) into (9.5) leads to

$$b^A_{(1)} = \tfrac{1}{2} \frac{dx^A}{ds} = W^A (c^2 \rho + \mu |h|^2) W^D f_{,D} - \mu (h^D f_{,D}) h^A \tag{9.6}$$

$$b_{(2)}^{A} \equiv \tfrac{1}{2}\,\frac{dx^{A}}{ds} \equiv \Big[2(c^{2}-2a^{2})\rho c^{2}(W^{D}f_{,D})^{2} + \Big\{(c^{2}-a^{2})\mu|h|^{2} + a^{2}c^{2}\rho\Big\}W^{D}f_{,D}(g^{AB}f_{,A}f_{,B})\Big]W^{A}$$

$$+ \Big[\Big\{(c^{2}-a^{2})\mu\ h^{2} + a^{2}c^{2}\rho\Big\}(W^{D}f_{,D})^{2} - a^{2}\mu(h^{D}f_{,D})^{2}\Big]\,g^{AB}f_{,B}$$

$$- a^{2}\mu(h^{D}f_{,D})(g^{AB}f_{,A}f_{,B})h^{A}. \qquad (9.7)$$

The corresponding differential equations for rays based on (5.3)', (6.7)' and (6.7)", can be read off from the above relations. The following qualitative discussion is the same for all such rays.

From (9.6) we observe that $\underline{b}_{(1)}$ lies in the plane of the orthogonal 4-vectors $\underline{W}$ and $\underline{h}$. The ray $\underline{b}_{(2)}$ will also lie in the same plane if $\underline{N}$ does. We now set out to investigate that possibility.

We have $\underline{h}/|h|$, as a unit space-like vector and $\underline{W}$, as a unit time-like vector. Let us introduce two new mutually orthogonal 4-vectors $\underline{Y}$ and $\underline{Z}$, which are orthogonal to the 2-base determined by W and $h/|h|$. Hence both $\underline{Y}$ and $\underline{Z}$ are space-like and span the null domain of H^{*AB} and therefore they span the non-null domain of H^{AB}. Thus we can decompose H^{AB}, in terms of Y^{A} and Z^{A}, getting (in view of the relations (2.10) and (2.11)):

$$H^{AB} = \mu\,|h|\,(Y^{A}Z^{B} - Z^{A}Y^{B}). \qquad (9.8)$$

The vectors $\underline{W}$, $\underline{h}/|h|$, $\underline{Y}$ and $\underline{Z}$ form a 4-tuple of mutually orthogonal unit vectors; W is time-like and ($\underline{h}/|h|$, $\underline{Y}$, $\underline{Z}$) are space-like. We can represent the components N_{A} as

$$N_{A} = AW_{A} + B\,h_{A}/|h| + C\,Y_{A} + D\,Z_{A}. \qquad (9.9)$$

Since $N_A N^A = -1$, this gives

$$A^2 - B^2 - C^2 - D^2 = -1. \tag{9.10}$$

Also

$$L = A, \qquad h_N = -B|h|. \tag{9.11}$$

The characteristic relations (5.3) and (6.7) become

$$(\mu|h|^2 + c^2\rho)A^2 - \mu|h|^2 B^2 = 0, \tag{9.12}$$

and

$$(c^2-2a^2)c^2\rho A^4 - \left\{(c^2-a^2)\mu|h|^2 + a^2c^2\rho\right\}A^2 + a^2\mu|h|^2B^2 = 0. \tag{9.13}$$

If the normal $\underline{N}$ is to lie in the 2-base determined by $\underline{W}$ and $\underline{h}$ then from (9.9) and (9.10) it follows that

$$C = D = 0, \tag{9.14}$$

and

$$A^2 - B^2 = -1\ . \tag{9.15}$$

Thus A and B satisfy the equation for a hyperbola as given by (9.15). The intersection of this hyperbola with the curve given by the equation (9.12) yields

$$A = (\mu/c^2\rho)^{\frac{1}{2}}\ |h|;\ B = \pm\left\{1 + \frac{\mu|h|^2}{c^2\rho}\right\}^{\frac{1}{2}}\ . \tag{9.16}$$

Similarly the intersection of this hyperbola with the curve given by (9.13), leads to

$$A = a \; / \; (c^2 - 2a^2)^{\frac{1}{2}} \quad , \; B = \pm \left\{ (c^2 - a^2) \; / \; (c^2 - 2a^2) \right\}^{\frac{1}{2}} ; \tag{9.17}$$

and the same roots as given by the relation (9.17) for Alfven waves. In fact the equation (9.12) becomes a factor of (9.13) when $B^2 = 1 + A^2$. From (9.16) it follows that there exist two wave normals which lie in the base ($\underline{W}$, $\underline{h}$) provided $c^2 > 2a^2$. The corresponding condition for Bruhat's analysis is $1 > \tilde{a}^2$.

Finally we observe from (9.10), (9.12) and (9.13) that $A = B = 0$; $C^2 + D^2 = 1$, satisfy these equations. Thus there are ∞' characteristic hypersurfaces such that $\underline{W}$ and $\underline{h}$ lie on their local tangent hyperplanes.

10. Geometry of the wave fronts.

Let us set

$$f_{,A} = X_A \, . \tag{10.1}$$

The equations (9.3) and (9.4) then become

$$F(X_A) = (c^2 \rho + \mu |h|^2)(W^A X_A)^2 - \mu (h^A X_A)^2 = 0. \tag{10.2}$$

$$F(X_A) = (c^2 - 2a^2) \rho c^2 (W^A X_A)^4 + \left\{ (c^2 - a^2) \mu \, |h|^2 + a^2 c^2 \rho \right\} (W^A X_A)^2 (g^{CD} X_C X_D)$$

$$- a^2 \mu (h^A X_A)^2 \; (g^{CD} X_C X_D) = 0. \tag{10.3}$$

The equation $F(X_A) = 0$, represents, for fixed (x, t), a surface in X-space called the surface of wave normals. Let us take the Lorentz frame at the point (x, t) such that, $g^{oo} = 1$, $g^{ij} = - \delta^{ij}$. Let us also define a 4-tuple of mutually orthogonal vectors $\underline{e}_o, \underline{e}_1, \underline{e}_2, \underline{e}_3$ at that point. For $\underline{e}_o$ and $\underline{e}_1$ we take $\underline{W}$ and $\underline{h}/|h|$ respectively. The equation (10.2) which gives the wave normal surface for Alfven waves becomes

$$(c^2\rho + p + \mu\,|h|^2)\,X_0^2 - \mu\,X_1^2 = 0. \tag{10.4}$$

Factoring the right member of the equation (10.4) we obtain two planes in the E-R space.

The equation (10.3), similarly, gives

$$\left\{(c^2 - a^2)(c^2\rho + \mu|h|^2)\right\} X_0^4 - \left[\left\{(c^2-a^2)\mu|h|^2 + a^2c^2\rho\right\}(X_1^2 + X_2^2 + X_3^2)\right.$$

$$\left. + a^2\mu\,|h|^2 X_1^2\right] X_0^2 + a^2\,\mu\,|h|^2 X_1^2\,(X_1^2 + X_2^2 + X_3^2) = 0, \tag{10.5}$$

which is tne equation for a conoid. The intersection of this conoid with a straight line shows that it is composed of two distinct nappes C_f (for fast wave) and C_s (for slow wave). The nappe C_f is interior and is convex. The light cone is interior to C_f. The two planes, which comprise the surface of the wave normals for the Alfven wave, always touch the conoid given by equation (10.5). The geometrical shape of these wave normal surfaces is thus the same as in the case of non-relativistic magnetohydrodynamics. Since the diagram of wave fronts can be constructed from the surfaces of wave normals, by a simple geometrical device, it follows that all the known geometrical facts about the magnetohydrodynamic wave fronts are applicable to the present case [8].

Finally we observe that when $c^2 = 2a^2$, the equation (10.3) has a factor

$$g^{CD}\,X_C X_D = 0, \tag{10.6}$$

which is tne equation of the light cone. In this case the light cone coincides with the nappe C_f of the fast wave.

This work was sponsored in part by the National Science Foundation under Grant No. GP-4026 and in part by NASA Grant No. NsG-134-61 with the Pennsylvania State University.

References

1. Akhieser, I. A., & Polovin, R. V., Theory of relativistic magnetohydrodynamic waves, JETP, Vol. 9, 1959, pp. 1316-1320.

2. Coburn, N., Discontinuity relations for charged, compressible, relativistic fluids, J. Math. Mech., Vol. 10, 1961, pp. 361-392.

3. Coburn, N. Intrinsic characteristic equations for relativistic compressible hydromagnetic flows, Buletinul Institutului Politechnic Din Iasi, Vol. 7(11), 1961, pp. 1-22.

4. Edlen, D. G. B., Material momentum-energy tensors and the calculus of variations, Proc. Nat. Acad. Sci. Vol. 51, 1964, pp. 367-372.

5. Foures-Bruhat, Y., Fluids relativistes de conductibilite infinie, Astronautica Acta, Vol. 6, 1960, pp. 354-365.

6. Giere, A., Magnetosonic wave velocities in a charged relativistic fluid, unpublished.

7. de-Hoffman, F. and Teller, E., Magnetohydrodynamic shocks, Phys. Rev., Ser. 2, Vol. 80, 1950, pp. 692-703.

8. Kanwal, R. P., Notes on Magnetohydrodynamics Part I, Scientific Report No. 218, Ionosphere Research Lab. Penn State University, September 1964.

9. Lichnerowicz, A., Theories relativistes de la gravitation et de l'electromagnetisme, Masson et Cie., Paris, 1955, p. 42.

10. Pham, M. Q., Etude electromagnetique et thermodynamique d'un fluid relativiste charge, J. Rat. Mech. Anal., Vo. 5, 1956, pp. 473-538.

11. Reichel, P., Basic notions of relativistic hydromagnetics, A.E.C. Computings and Applied Math. Center, Institute of Mathematical Sciences, N.Y. University, 1956.

12. Saini, G. L., Singular hypersurfaces of order one in relativistic magneto-fluid-dynamics, Proc. Roy. Soc. A., Vol. 260, 1961, pp. 61-78.

13. Schouten, J. A., Ricci Calculus, Springer Verlag, Berlin, 1954, pp. 35-36.

14. Taub, A. H., Hydrodynamics and General Relativity, Proc. Sympos. Michigan State University, Academic Press, 1963, pp. 21-28.

15. Thomas, T. Y., The perfect gas, Proc. Nat. Acad. Sc., Vol. 51, 1964, pp. 363-367.

61. Zumino, B., Some questions in relativistic hydromagnetics, Phys. Rev., Vol. 108, 1957, pp. 1116-1121.

The Ionosphere Research Laboratory and
The Department of Mathematics
The Pennsylvania State University
University Park, Pennsylvania.